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2001 J. Phys. A: Math. Gen. 34 2097

(http://iopscience.iop.org/0305-4470/34/11/303)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 2097–2109 www.iop.org/Journals/ja PII: S0305-4470(01)17431-7

Spinning gas clouds: Liouville integrability

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Received 26 September 2000, in final form 11 January 2001

Abstract

This paper constitutes a generalization to arbitrary states of rotation of an earlier work in which we showed Liouville integrability of the expanding and rotating gas cloud model of Ovsiannikov and of Dyson in cases of rotation around a fixed principal axis.

PACS numbers: 0230J, 0220S

1. Introduction

Recently (Gaffet 2000a) the spinning cloud model of Ovsiannikov (1956) and Dyson (1968) has been shown to be a completely integrable Hamiltonian system (in the Liouville sense), under the restricting conditions of rotation of the cloud around a fixed principal axis and of zero vorticity.

In this paper we extend this result to all cases of rotation without vorticity, through the introduction of two new integrals of the motion: I_6, which generalizes a corresponding integral found in our earlier study; and an altogether new integral, L_6, which identically vanishes in cases of rotation around a fixed axis.

Let us recall briefly that the model of Ovsiannikov and Dyson constitutes an ordinary differential reduction of the equations of gas dynamics, characterized by a linear relation between Eulerian ($\vec{x}$) and Lagrangian coordinates ($\vec{\alpha}$):

$$x_i = F_{ij}(t)\alpha_j \tag{1.1}$$

and a Gaussian distribution of density. It describes an expanding and rotating gas cloud (which we take to be monatomic, with adiabatic index $\gamma = 5/3$) of ellipsoidal shape, with principal axes of length D_1, D_2 and D_3.

In section 2 we introduce a new formalism whose main advantage is its invariance under the group of permutations of the ellipsoid's axes. The basic variables are then the permutation invariants constructed by forming simple combinations of the matrix F and of $V \equiv \dot{F}F^{-1}$, which is the matrix describing the velocity distribution $\vec{v}(\vec{x})$ in a Eulerian framework:

$$v_i = V_{ij}x_j. \tag{1.2}$$

In sections 3 and 4 we determine the integrals of motion in the two limits where the matrix V is either large or small; the usefulness of these results arises from the fact that the highest- and lowest-degree terms (in powers of V) in any integral of motion must be integrals of these two limiting cases, respectively.

In section 5 we construct the generalized integral I_6, which is found to admit a particularly simple expression in this permutation invariant formalism, and the algebraic constraint ($\det \mathrm{D} = 1$) is reformulated in terms of permutation invariant variables; this turns out to be a crucial step toward the determination of the integral L_6.

In section 6 the last missing integral L_6 is constructed. Its leading term curiously involves a three-vector $\vec{\tilde{j}}$, which is associated with the angular momentum vector $\vec{j}$ through a symmetry that inverses the lengths of the principal axes, while preserving the traceless part of the matrix V.

2. A permutation invariant formalism

2.1. The new system of variables

We start with the equations of motion, as derived in a recent work (Gaffet 2000b):

$$\frac{\mathrm{d}v}{\mathrm{d}u} + v^2 - \frac{1}{3}\,\mathrm{Tr}(v^2) = [\omega; v] + \mathrm{D}^{-2} - \frac{1}{3}\,\mathrm{Tr}(\mathrm{D}^{-2}) \tag{2.1}$$

where the symbol $[\omega; v]$ denotes the commutator of ω and v, the independent variable u is the integral of temperature (T_e) over time,

$$u = \int T_e\,\mathrm{d}t \tag{2.2}$$

v is the 3×3 symmetric matrix which is obtained by taking the traceless part, $V - \frac{1}{3}\,\mathrm{Tr}(V)$ of $V \equiv \dot{F}F^{-1}$, transforming it to the moving frame through the appropriate rotation and finally rescaling it by a factor T_e. The matrix $\mathrm{D} = \mathrm{diag}(\mathrm{D}_1, \mathrm{D}_2, \mathrm{D}_3)$ has been rescaled too (by a factor $\sqrt{T_e}$) in such a way that D has unit determinant:

$$\det(\mathrm{D}) \equiv \mathrm{D}_1\mathrm{D}_2\mathrm{D}_3 = 1. \tag{2.3}$$

The antisymmetric matrix ω (which represents the angular velocity of the cloud) is fixed by the off-diagonal elements of v:

$$\omega_{23} = \frac{I_1}{\hat{I}_1} v_{23} \tag{2.4}$$

where

$$I_1 \equiv \mathrm{D}_2^2 + \mathrm{D}_3^2$$
$$\hat{I}_1 \equiv \mathrm{D}_2^2 - \mathrm{D}_3^2$$

together with the equations deducible by circular permutation of the indices.

The system must be completed by the equations of evolution of D:

$$\frac{\mathrm{d}}{\mathrm{d}u}\ell n\mathrm{D}_1 = v_{11} \quad \text{(and circular permutation).} \tag{2.5}$$

To obtain a permutation invariant formulation of the system (2.1), (2.5) we simply start with the two variables X_0, Y_0:

$$\begin{aligned} X_0 &= \mathrm{Tr}(\mathrm{D}^2) \\ Y_0 &= \mathrm{Tr}(\mathrm{D}^{-2}) \end{aligned} \tag{2.6}$$

and differentiate several times in sequence, until a closed system of differential equations has been obtained. In the process, new auxiliary variables X_1, X_2, Y_1, Y_2, T and P have been introduced, and the system reads

$$T'(u) = 3P - Y_1 \tag{2.7a}$$

$$P'(u) = -\tfrac{2}{3}T^2 + (\tfrac{2}{3}TY_0 + Y_2) \tag{2.7b}$$

$$X_0'(u) = 2X_1 \tag{2.8a}$$

$$X_1'(u) = \left(X_2 - \frac{2}{3}TX_0\right) + \left(3 - \frac{X_0Y_0}{3}\right) \tag{2.8b}$$

$$X_2'(u) = -\tfrac{4}{3}TX_1 - \tfrac{2}{3}Y_0X_1 \tag{2.8c}$$

$$Y_0'(u) = -2Y_1 \tag{2.9a}$$

$$Y_1'(u) = -\left(3Y_2 + \frac{2}{3}TY_0\right) + 2\left(\frac{Y_0^2}{3} - X_0\right) \tag{2.9b}$$

$$Y_2'(u) = 4(\tfrac{2}{3}TY_1 - PY_0) + 2(\tfrac{2}{3}Y_0Y_1 + X_1). \tag{2.9c}$$

The equations (2.7a) and (2.7b) determine the evolution of the variables T and P, which are in fact the coefficients of the characteristic equation of the matrix v:

$$v^3 + Tv - P = 0. \tag{2.10}$$

The equations (2.8) and (2.9) similarly define the evolution of six variables X_0, X_1, X_2, Y_0, Y_1 and Y_2, which may be identified with

$$\begin{cases} X_n = \mathrm{Tr}(\mathrm{D}v^n\mathrm{D}) \\ Y_n = \mathrm{Tr}(\mathrm{D}^{-1}v^n\mathrm{D}^{-1}) \end{cases} \qquad (n = 0, 1, 2). \tag{2.11}$$

2.2. Inverse transformation formulae

In order to express the quantities of interest, such as the total energy and angular momentum in terms of T, P, X_n and Y_n we need to invert the transformation formulae that give these new variables in terms of the matrices D and v.

The determination of D merely involves the two variables X_0 and Y_0, as the three eigenvalues of $\Delta \equiv \mathrm{D}^2$ are the roots of the characteristic equation:

$$\Delta^3 - X_0\Delta^2 + Y_0\Delta - 1 = 0. \tag{2.12}$$

Next, to obtain the diagonal part of v, we only need to consider the two variables X_1 and Y_1 in addition to the elements of Δ:

$$v_{11} = \frac{-(\Delta_1X_1 + Y_1)}{\hat{I}_2\hat{I}_3} \tag{2.13}$$

(and circular permutation of the indices—except, of course, the indices of X and Y). Lastly, the off-diagonal elements of v are determined by their squares:

$$\hat{I}_2\hat{I}_3v_{23}^2 = I_2I_3(T - T_0) + \Delta_1(X_2 - X_{20}) + (Y_2 - Y_{20}) \tag{2.14}$$

where T_0, X_{20} and Y_{20} are quadratic combinations of X_1 and Y_1:

$$\begin{aligned} -2hT_0 &\equiv \Sigma_2X_1^2 + 2\Sigma_1X_1Y_1 + \Sigma_0Y_1^2 \\ hX_{20} &\equiv \Sigma_3X_1^2 + 2\Sigma_2X_1Y_1 + \Sigma_1Y_1^2 \\ hY_{20} &\equiv \Sigma_1X_1^2 + 2\Sigma_0X_1Y_1 + \Sigma_{-1}Y_1^2 \end{aligned} \tag{2.15}$$

$$h = (\hat{I}_1\hat{I}_2\hat{I}_3)^2 \tag{2.16}$$

and

$$\Sigma_n = \sum_i (\Delta_i^n\hat{I}_i^2). \tag{2.17}$$

The $\sum_n$ and h are polynomial functions of X_0 and Y_0:

$$\begin{aligned}\Sigma_0 &= 2(X_0^2 - 3Y_0)\\ \Sigma_1 &= (X_0Y_0 - 9)\\ \Sigma_2 &= 2(Y_0^2 - 3X_0)\end{aligned} \tag{2.18}$$

and

$$3h = (\Sigma_0\Sigma_2 - \Sigma_1^2). \tag{2.19}$$

Other sums $\sum_n$ may be easily found through the recursion relation:

$$\Sigma_{n+3} = X_0\Sigma_{n+2} - Y_0\Sigma_{n+1} + \Sigma_n. \tag{2.20}$$

It is worth noting that the symmetry that exchanges X_0 and Y_0 changes $\sum_{1+n}$ to $\sum_{1-n}$.

Let us finally remark that the constraint $\det\Delta = 1$ (equation (2.3)) must translate into a corresponding algebraic constraint relating the eight new variables; that question will be examined later (section 5).

2.3. The energy constant

As is well known, the gas cloud model of Ovsiannikov and of Dyson may be viewed equivalently as representing Hamiltonian motion of a point mass in nine-dimensional Euclidean space (the space of Cartesian coordinates F_{ij}); and, when the gas is monatomic—as assumed here—the radial motion may be separated out, resulting in a Hamiltonian motion on the eight-dimensional unit hypersphere. The energy constant of that hyperspherical motion, denoted $\hat{E}$ in our earlier works, is a combination of the energy E of the original motion (in nine-dimensional flat space) and of two additional constants (Anisimov and Lysikov 1970). Its expression in terms of the new variables is particularly simple:

$$2\hat{E} = (X_0X_2 - X_1^2) + 3X_0 \tag{2.21}$$

where the first part (in parentheses) represents kinetic energy, and the last term, $3X_0$, thermal (or potential) energy.

2.4. The angular momentum

The angular momentum vector $\vec{j}$ (in the moving frame) is related to the angular velocity vector $\vec{\omega}$, in the absence of vorticity, as (Gaffet 2000a, b)

$$j_1 = \frac{\hat{I}_1^2}{I_1}\omega_1 \quad \text{(and circular permutation)} \tag{2.22}$$

or, directly in terms of the off-diagonal elements of v, as (see equation (2.4), where $\omega_{23} = \omega_1$ etc)

$$j_1 = \hat{I}_1 v_{23} \quad \text{(and circular permutation).} \tag{2.23}$$

The three components of the angular momentum $\vec{J}$ in the fixed frame remain, of course, constant; the total angular momentum, $\vec{J}^2 = \vec{j}^2$, is also constant, and is expressed in a permutation invariant way by the following simple formula:

$$\vec{J}^2 = (X_0X_2 - X_1^2) + (3Y_2 + 4TY_0). \tag{2.24}$$

The first group of terms on the right-hand side is just the kinetic energy (see equation (2.21)) of hyperspherical motion, so we can rewrite the constant of the motion in an even simpler form:

$$\tfrac{1}{3}(\vec{J}^2 - 2\hat{E}) = (\tfrac{4}{3}TY_0 + Y_2) - X_0. \tag{2.25}$$

3. The integrals of hyperspherical geodesic motion

Each of the eight new variables is a homogeneous function of the matrix v, characterized by a certain degree: T is of degree two, P of degree three and X_n and Y_n both of degree n. The derivative of any variable (of degree n), as given by the equations of motion (2.7)–(2.9), contains two parts: a first one of degree $n+1$, and another of degree $n-1$. These two parts manifestly represent the asymptotic forms of the system in the two limits of large and of low velocity, respectively. As the lower-degree terms represent the dynamical effect of thermal pressure inside the cloud, while the higher-degree ones represent purely inertial effects, the large-velocity limit may be viewed as describing hyperspherical geodesic motion. In that limit the system (2.7)–(2.9) reduces to

$$T'(u) = 3P \tag{3.1a}$$
$$P'(u) = -\tfrac{2}{3}T^2 \tag{3.1b}$$

$$X_0'(u) = 2X_1 \tag{3.2a}$$
$$X_1'(u) = (X_2 - \tfrac{2}{3}TX_0) \tag{3.2b}$$
$$X_2'(u) = -\tfrac{4}{3}TX_1 \tag{3.2c}$$

$$Y_0'(u) = -2Y_1 \tag{3.3a}$$
$$Y_1'(u) = -(3Y_2 + \tfrac{2}{3}TY_0) \tag{3.3b}$$
$$Y_2'(u) = 4(\tfrac{2}{3}TY_1 - PY_0). \tag{3.3c}$$

The first thing to be remarked is that equations (3.1a) and (3.1b) constitute a closed sub-system for the variables T and P; that the equations (3.2) are then a linear system for the X_i and the equations (3.3) another (independent) linear system for the Y_i.

The closed sub-system has an obvious first integral,

$$I_6^6 = 27P^2 + 4T^3 \tag{3.4}$$

which is the discriminant of the characteristic equation of v (equation (2.10)). The integrals of the linear sub-systems must be homogeneous functions of X_n (respectively Y_n); each sub-system possesses its full complement of first integrals: one linear, one quadratic and one cubic.

3.1. The integrals of the linear system for X

The integral linear in X is of degree four in v:

$$\varepsilon_4 = 3TX_2 - 9PX_1 - T^2X_0. \tag{3.5}$$

The integral of degree two in X,

$$\varepsilon_2 = (X_0X_2 - X_1^2) \tag{3.6}$$

is quadratic in v, and in fact is the kinetic energy (see equation (2.21)), as expected. Finally, an integral cubic in X, and of degree six in v, denoted ε_6, will be found in section 5 as a by-product of the determination of the form of the algebraic constraint.

3.2. The integrals of the linear system for Y

The integral linear in Y (degree two in v) reads

$$\eta_2 = Y_2 + \tfrac{4}{3}TY_0 \tag{3.7}$$

and coincides with the leading term of the exact integral (2.25), as expected. There exists an integral quadratic in Y (degree 10 in v), but its expression is not particularly simple and it is probably not of interest in the present problem.

The integral cubic in Y, degree six in v, will also be presented in section 5.

3.3. A symmetry relating the linear systems for X and Y

Let us consider the following variable:

$$Z = 3X_2 + TX_0. \tag{3.8}$$

By differentiation we obtain $Z'(u) = 3PX_0 - 2TX_1$, and

$$Z''(u) = -\tfrac{2}{3}TZ. \tag{3.9}$$

Starting now with the variable

$$\tilde{Z} = 3PY_0 + TY_1 \tag{3.10}$$

we find

$$\tilde{Z}'(u) = -(\tfrac{8}{3}T^2Y_0 + 3PY_1 + 3TY_2)$$

and then

$$\tilde{Z}''(u) = -\tfrac{2}{3}T\tilde{Z}$$

i.e. $\tilde{Z}$ satisfies the same second-order o.d.e. (equation (3.9)) as Z does.

In particular, the Wronskian w_6 constructed from Z and $\tilde{Z}$ must be an integral of free (geodesic) motion:

$$w_6 = (\tilde{Z}Z' - Z\tilde{Z}') \tag{3.11}$$

w_6 is algebraically related to the already determined first integrals, which form a complete set.

Let us mention the existence of further new integrals, cubic in Z:

$$(PZ^3 - TZ^2Z' - Z'^3)$$

and

$$\{(27P^2 + 2T^3)Z^3 - 27PTZ^2Z' + 18T^2ZZ'^2 + 27PZ'^3\}.$$

Owing to the symmetry of the roles of Z and $\tilde{Z}$, cubic integrals in $\tilde{Z}$ of exactly the same form also exist.

4. The low-velocity limit

Let us now consider the system in the opposite limit of zero velocity:

$$T'(u) = -Y_1 \tag{4.1a}$$

$$P'(u) = (\tfrac{2}{3}TY_0 + Y_2) \tag{4.1b}$$

$$X_0'(u) = 0 \tag{4.2a}$$
$$X_1'(u) = \left(3 - \frac{X_0 Y_0}{3}\right) \tag{4.2b}$$
$$X_2'(u) = -\tfrac{2}{3} Y_0 X_1 \tag{4.2c}$$

$$Y_0'(u) = 0 \tag{4.3a}$$
$$Y_1'(u) = 2\left(\frac{Y_0^2}{3} - X_0\right) \tag{4.3b}$$
$$Y_2'(u) = 2(\tfrac{2}{3} Y_0 Y_1 + X_1). \tag{4.3c}$$

X_0 and Y_0 are obvious constants of the motion. The three components of angular momentum ($\vec{j}$) in the moving frame are also constant—since their evolution is governed by

$$\frac{\mathrm{d}\vec{j}}{\mathrm{d}u} = \vec{j} \wedge \vec{\omega} \tag{4.4}$$

where the right-hand side, of purely inertial origin, vanishes in the present limit. In other words, the matrix D and the off-diagonal part of v are both constants of the motion.

This suggests introducing a transformation, here denoted $(\tilde{T})$, which consists of the inversion of D, without changing v: $(\tilde{T})$ exchanges the X_n and Y_n without affecting T and P; and transforms (see equations (2.3) and (2.23)) the angular momentum $\vec{j}$ into a new constant vector $\vec{\tilde{j}}$:

$$\tilde{j}_1 = -\Delta_1 j_1 \quad \text{(and circular permutation).} \tag{4.5}$$

One may further introduce a corresponding vector $\vec{\tilde{J}}$ in the fixed frame—related to $\vec{\tilde{j}}$ by the same rotation that relates $\vec{J}$ and $\vec{j}$.

The three constants of angular momentum may then be conveniently represented by

$$\begin{aligned} \vec{J}^2 &= (X_0 X_2 - X_1^2) + (3Y_2 + 4T Y_0) \\ -\vec{J} \cdot \vec{\tilde{J}} &= (X_0 Y_0 + 3)T + (X_0 Y_2 + Y_0 X_2) + X_1 Y_1 \\ \vec{\tilde{J}}^2 &= (Y_0 Y_2 - Y_1^2) + (3X_2 + 4T X_0). \end{aligned} \tag{4.6}$$

In view of the formula (2.14) determining the off-diagonal elements of v, these three constants must be linear combinations, with constant coefficients, of $(T - T_0)$, $(X_2 - X_{20})$ and $(Y_2 - Y_{20})$; therefore they are linear homogeneous combinations of T, X_2, Y_2, X_1^2, $X_1 Y_1$ and Y_1^2.

Still another constant is the Wronskian constructed from X_1 and Y_1:

$$\begin{aligned} W_1 &= (X_1 Y_1' - Y_1 X_1') \\ &= \tfrac{1}{3}(\Sigma_2 X_1 + \Sigma_1 Y_1). \end{aligned} \tag{4.7}$$

5. The algebraic constraint

As mentioned above, the constraint (2.3) that D should have unit determinant translates into an algebraic constraint relating the new variables T, P, X_n and Y_n. Before discussing this, let us now introduce a new exact integral of the motion, based on the results of the preceding sections.

5.1. The new integral I_6

The first integral of free motion I_6^6 (equation (3.4)) turns out to be the leading term of an exact integral, I_6, of the complete system (2.7)–(2.9), including the effect of thermal pressure. The following term in the expansion, I_6^4, quartic in v, is given by the equation

$$-\mathrm{d}_F I_6^4 = \frac{\mathrm{d}}{\mathrm{d}u} I_6^6 = 6[6PTY_0 - 2T^2Y_1 + 9PY_2] \qquad (5.1)$$

where d_F means the derivative $\mathrm{d}/\mathrm{d}u$ as defined by equations (3.1)–(3.3), in the absence of pressure forces. As the operation of d_F on a monomial in X and Y (with coefficient function of T and P) produces a homogeneous polynomial of the same degree in X, and also in Y, it is clear that I_6^4 must be linear homogeneous in Y, and independent of X. Further, being quartic in v, it must be a linear combination of the following three terms: T^2Y_0, PY_1, TY_2; in this way we obtain

$$I_6^4 = 6[4T^2Y_0 + 9PY_1 + 6TY_2]. \qquad (5.2)$$

The next term in the expansion is given by

$$-\mathrm{d}_F I_6^2 = \frac{\mathrm{d}}{\mathrm{d}u}(I_6^6 + I_6^4) = 36\left[\frac{Y_1Y_2}{2} + T(Y_0Y_1 + 2X_1) + P(Y_0^2 - 3X_0)\right]. \qquad (5.3)$$

By the same reasoning, I_6^2 must be a sum of monomials of the types $Y_i\ Y_j$ and X_i, with coefficients that are functions of T and P; and, being quadratic in v, there are only five possible terms:

$$TX_0 \qquad X_2 \qquad TY_0^2 \qquad Y_0Y_2 \qquad Y_1^2.$$

The resulting expression,

$$\frac{I_6^2}{36} = \left(Y_0Y_2 - \frac{Y_1^2}{4} + 3X_2\right) + T(Y_0^2 + X_0), \qquad (5.4)$$

is an integral of the limit of low-velocity motion (section 4); therefore, the sum $I_6^6 + I_6^4 + I_6^2$ constitutes an exact integral of the system (2.7)–(2.9).

Let us point out that I_6^2, being linear homogeneous in T, X_2, Y_2 and Y_1^2, is of the same general type as the angular momentum integrals (4.6), and in fact is a linear combination of them and of the square of the Wronskian W_1 (see equation (4.7)):

$$-\frac{h}{9}I_6^2 = \Sigma_2^2\vec{J}^2 + 2\Sigma_1\Sigma_2\vec{J}\cdot\vec{\tilde{J}} + \frac{(4\Sigma_1^2 - \Sigma_0\Sigma_2)}{3}\vec{\tilde{J}}^2 + 9W_1^2 = (\Sigma_2\vec{J} + \Sigma_1\vec{\tilde{J}})^2 - h\vec{\tilde{J}}^2 + 9W_1^2. \qquad (5.5)$$

5.2. The algebraic constraint

The most direct way to derive this is to observe that the product $P_1 \equiv v_{12}v_{23}v_{31}$ of the off-diagonal elements of v is given rationally by the general formula

$$2P_1 = (P - P_0) + Q_1 \qquad (5.6)$$

where $P_0 \equiv v_{11}v_{22}v_{33}$ is the product of diagonal elements, and $Q_1 \equiv v_{11}v_{23}^2+$ (circular permutation).

On the other hand, the square P_1^2 of P_1 may be independently found by means of the formula (2.14), which gives the squares of the off-diagonal elements. The algebraic constraint thus reads

$$(P_1)^2 = P_1^2. \tag{5.7}$$

We obtain—based on the inverse transformation formulae (2.13)–(2.15)—the following expression for P_1:

$$\begin{aligned}
-2hP_1 = {} & -h(P - P_0) + (SX_1 + \tilde{S}Y_1)(T - T_0) \\
& +(\Sigma_2 X_1 + \Sigma_1 Y_1)(X_2 - X_{20}) + (\Sigma_1 X_1 + \Sigma_0 Y_1)(Y_2 - Y_{20}) \\
= {} & -hP + T(SX_1 + \tilde{S}Y_1) + X_2(\Sigma_2 X_1 + \Sigma_1 Y_1) + Y_2(\Sigma_1 X_1 + \Sigma_0 Y_1) \\
& +2[X_1^3 + X_1^2 Y_0 Y_1 + X_0 X_1 Y_1^2 + Y_1^3]
\end{aligned} \tag{5.8}$$

where

$$\begin{aligned}
S &= (\Sigma_0 + X_0 \Sigma_2) \\
\tilde{S} &= (\Sigma_2 + Y_0 \Sigma_0)
\end{aligned} \tag{5.9}$$

while P_1^2 is given by

$$\begin{aligned}
hP_1^2 = {} & (X_0 Y_0 - 1)^2 (T - T_0)^3 + 2(X_0 Y_0 - 1)(T - T_0)^2 [Y_0(X_2 - X_{20}) + X_0(Y_2 - Y_{20})] \\
& +(T - T_0)[(Y_0^2 + X_0)(X_2 - X_{20})^2 + 3(X_0 Y_0 - 1)(X_2 - X_{20})(Y_2 - Y_{20}) \\
& +(X_0^2 + Y_0)(Y_2 - Y_{20})^2] + [(X_2 - X_{20})^3 + Y_0(X_2 - X_{20})^2(Y_2 - Y_{20}) \\
& +X_0(X_2 - X_{20})(Y_2 - Y_{20})^2 + (Y_2 - Y_{20})^3].
\end{aligned} \tag{5.10}$$

Separating the terms of various types according to their degrees in X and in Y, the relation (5.7) assumes the following form:

$$h[P_1^2 - (P_1)^2] \equiv \theta_6 + \varepsilon_6 + \tilde{\varepsilon}_6 + \zeta_6 + \frac{I_6^6}{4} = 0 \tag{5.11}$$

where θ_6 is a sum of monomial terms of the type $X_i X_j Y_k Y_l$; ε_6 and $\tilde{\varepsilon}_6$ are respectively of the types $X_i X_j X_k$ and $Y_i Y_j Y_k$; ζ_6 is composed of terms $X_i Y_j$, and I_6^6 is, of course, independent of X and Y. All terms are of degree six in v.

Let us first remark that the identity (5.11) has to be symmetric under the transformation $(\tilde{T})$ introduced in section 4—since the inversion of D preserves the constraint (2.3) of unimodularity. Therefore, θ_6 and ζ_6 are both symmetric, and $\tilde{\varepsilon}_6$ is the symmetric counterpart of ε_6. Further, as the identity obviously remains valid in the limit of free motion, where the governing equations (see section 3) are homogeneous in X and in Y, each of the five terms on the right-hand side of equation (5.11), being characterized by a different degree (in X and in Y) from that of the other terms, must be an integral of free motion, as is the last term, I_6^6. This is indeed true of the fourth term, which may be identified with the following combination of integrals:

$$\zeta_6 \equiv \tfrac{1}{2}(\eta_2 \varepsilon_4 - w_6). \tag{5.12}$$

The existence of the integral ε_6, cubic in X, was mentioned in section 3.1:

$$\varepsilon_6 = P^2 X_0^3 + P X_1 (X_1^2 - 3X_0 X_2 - 2T X_0^2) + (T X_0 + X_2)(T X_1^2 + X_2^2). \tag{5.13}$$

The integral $\tilde{\varepsilon}_6$ has exactly the same form, after exchanging the roles of the X_n and Y_n.

The integral θ_6 involves a relatively large number of terms:

$$\theta_6 = -\frac{P^2}{4} X_0^2 Y_0^2 + P(X_0 Y_1 + Y_0 X_1)[T X_0 Y_0 + (X_0 Y_2 + Y_0 X_2) + X_1 Y_1]$$

$$-\frac{P}{2}X_0Y_0(X_1Y_2+X_2Y_1)+\{T^3X_0^2Y_0^2+2T^2X_0Y_0(X_0Y_2+Y_0X_2)$$
$$+T(X_0^2Y_2^2+3X_0X_2Y_0Y_2+X_2^2Y_0^2)+X_2Y_2(X_0Y_2+Y_0X_2)\}$$
$$+\{T^2X_0X_1Y_0Y_1+TX_1Y_1(X_0Y_2+Y_0X_2)-\tfrac{1}{4}(X_1Y_2-X_2Y_1)^2\}. \quad (5.14)$$

6. The last integral

6.1. The triple product $(\vec{j};\, v\vec{j};\, v^2\vec{j})$

Knowing now the seven integrals: $\hat{E}$, $\vec{J}^2$, J_3, K_1, K_2, K_3 and I_6 (where the K_i are the (zero) components of vorticity), one more integral is still needed to ensure Liouville integrability. In a recent article (Gaffet 2000b) I conjectured that it might have as leading term the triple product

$$\Phi \equiv (\vec{j};\, v\vec{j};\, v^2\vec{j})$$

which has the desired properties of first, being an integral of free motion (as shown in the above-mentioned work) and, in addition, of vanishing identically in cases of rotation around a principal axis. This motivated an attempt at determining the expression of this triple product Φ, in terms of the new set of variables T, P, X and Y. The form of the result turns out to be intimately related to that of the algebraic constraint discussed in the preceding section:

$$\Phi = \varepsilon_6 + \zeta_6 + \frac{I_6^6}{2}. \quad (6.1)$$

Unfortunately, it appears that there exists no exact integral of motion whose leading term coincides with the above expression: direct calculation of the next term (fourth degree in v), following the method of section (5.1), leads to an over-determined system of equations whose compatibility conditions are not satisfied (under the assumption that the sought for integral admits a polynomial formulation).

6.2. The last missing integral

The analysis of the preceding section 5 suggests a way out of this difficulty: to start from $\tilde{\Phi} \equiv (\vec{\tilde{j}};\, v\vec{\tilde{j}};\, v^2\vec{\tilde{j}})$ as leading term instead of Φ, in view of the fact that $\tilde{\Phi}$ shares with Φ the two fundamental properties of being an integral of free motion—since

$$\tilde{\Phi} \equiv \tilde{\varepsilon}_6 + \zeta_6 + \frac{I_6^6}{2} \quad (6.2)$$

and of vanishing in block-diagonal cases, since $\vec{\tilde{j}}$ then is, as well as $\vec{j}$, an eigenvector of v. (Let us remark here that this entails the relations $\varepsilon_6 = \tilde{\varepsilon}_6$, and $\theta_6 = \zeta_6 + \frac{3}{4}I_6^6$ in such cases.)

Let us then look for an exact integral whose leading term is $L_6^6 = \tilde{\Phi}$, i.e.

$$L_6^6 = \tilde{\varepsilon}_6 + \zeta_6 + \frac{I_6^6}{2}. \quad (6.3)$$

The next term in its expansion, L_6^4, of degree four in v, is given by (see section 5.1)

$$-d_F(L_6^4) = \frac{\mathrm{d}}{\mathrm{d}u}(L_6^6)$$
$$= \frac{\mathrm{d}}{\mathrm{d}u}(\tilde{\varepsilon}_6) + \frac{\mathrm{d}}{\mathrm{d}u}(\zeta_6) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}u}(I_6^6) \quad (6.4)$$

where $\frac{\mathrm{d}}{\mathrm{d}u}(I_6^6)$ has already been calculated (see equation (5.1)), and

$$\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}u}(\zeta_6) = \{3P(X_0X_2 - X_1^2) + 2PTX_0^2 - 2TX_1(TX_0 + X_2)\} + \{T^2Y_1 - 3PTY_0 - \tfrac{9}{2}PY_2\} + \Big\{PX_0Y_1^2 - \tfrac{5}{2}PX_0Y_0Y_2 - PX_2Y_0^2 - \frac{7}{3}PTX_0Y_0^2 + \frac{T^2}{3}X_0Y_0Y_1 + \frac{T}{3}X_0Y_1Y_2 - \tfrac{2}{3}TX_1Y_1^2 - \tfrac{4}{3}TX_1Y_0Y_2 - TX_2Y_0Y_1 - \tfrac{3}{2}X_1Y_2^2 - \tfrac{1}{2}X_2Y_1Y_2\Big\} \tag{6.5}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}u}(\tilde{\varepsilon}_6) = 2TX_0Y_0(PY_0 - TY_1) + 3P(X_0Y_0Y_2 - X_1Y_0Y_1 - X_0Y_1^2) + 2TY_2(Y_0X_1 - X_0Y_1) + TX_1Y_1^2 + 3X_1Y_2^2. \tag{6.6}$$

The equation (6.4) defining L_6^4 does admit a solution of polynomial form:

$$L_6^4 = \frac{I_6^4}{4} - 3\{T(X_1^2 + 2X_0X_2) + 3(X_2^2 - PX_0X_1)\} - 3\Big\{PY_0(X_1Y_0 + \tfrac{1}{2}X_0Y_1) + 2T^2X_0Y_0^2 + TY_0(X_1Y_1 + 2X_2Y_0 + 3X_0Y_2) + Y_2(2X_2Y_0 + X_0Y_2) + \frac{Y_1}{2}(X_1Y_2 - X_2Y_1)\Big\}. \tag{6.7}$$

Furthermore, the above L_6^4 turns out to be an integral of the system in its low-velocity limit (see section 4); therefore $L_6^6 + L_6^4$ is an exact integral of the complete system (2.7)–(2.9), including the effect of pressure.

7. Conclusion

The present reformulation of the problem of Ovsiannikov and Dyson, in terms of a set of permutation invariant variables, leads to a differential system (equations (2.7)–(2.9)) which possesses the Riccati-like property that the derivative of any variable is an at most quadratic function of all.

In this formulation, the algebraic constraint ($\det \mathrm{D} = 1$) assumes a non-trivial form; it is composed of several (five) separately homogeneous parts that, owing to the form of the differential system, must be integrals of free motion, that is, integrals of the limiting form of the system at high velocities. This turned out to be the key to the determination of the two integrals which were needed for Liouville integrability. One of these five homogeneous parts (see equation (5.11)) indeed constitutes the leading term (in the limit of high velocity) of a new integral I_6, while the sum of two other parts ($\tilde{\varepsilon}_6$ and ζ_6) leads to the integral L_6.

Another important property of the algebraic constraint is its symmetry under inversion of the diagonal matrix D, i.e. the transformation (denoted $(\tilde{T})$) that exchanges the X_i and the Y_i. This accounts for the existence of $(\tilde{T})$ symmetrical integrals of free motion, such as θ_6 and ζ_6, and of a $(\tilde{T})$ symmetrical pair of integrals (ε_6 and $\tilde{\varepsilon}_6$), in spite of the lack of symmetry of the equations of motion themselves in the high-v limit.

As a result, the vector $\vec{\tilde{j}}$ which is associated with the angular momentum vector $\vec{j}$ by the symmetry turns out to play an important role (as seen in particular in section 4: see e.g. equation (4.6)). Together with the decomposition into five distinct homogeneous terms of the algebraic constraint, the consideration of this new vector $\vec{\tilde{j}}$ was the key to the identification, performed in section 6, of the leading term of the last integral, which is just the triple product $(\vec{\tilde{j}};\, v\vec{\tilde{j}};\, v^2\vec{\tilde{j}})$.

Appendix A. Explicit Hamiltonian formulation

The equations of motion (2.1) considered in the present paper have been derived in an earlier work (Gaffet 2000b) from Dyson's equation:

$$F_T \ddot{F} = T_e \tag{A.1}$$

which represents Hamiltonian motion of a point mass in a potential in nine-dimensional Euclidean space (Dyson 1968). Our system (2.1) represents however a distinct Hamiltonian motion, which takes place in the curved eight-dimensional space $\mathrm{O}(3) \times S_2 \times \mathrm{O}(3)$, where the two O(3) are the rotation groups spanned by the rotation matrices O_1 and O_2 (in that order), and S_2 is the unit two-sphere, here parametrized by the diagonal unimodular matrix D (or equivalently, by the two coordinates X_0 and Y_0).

In the present problem, where there is no vorticity, the time evolution of O_2 (the matrix of rotation in Lagrangian space) is entirely deducible from that of O_1 (the matrix representing the orientation of the ellipsoidal cloud in ordinary space); as a result the system (2.1) turns out, as we show below, to describe a simpler Hamiltonian motion, which takes place in the five-dimensional space $\mathrm{O}(3) \times S_2$.

We expect the motion to be derivable from a Lagrangian which is the difference between kinetic and potential energies. The position of the point mass on the sphere S_2 being represented by the unit vector $\vec{r} \equiv \vec{\mathrm{D}}/\sqrt{X_0}$ (where $\vec{\mathrm{D}} \equiv (\mathrm{D}_1, \mathrm{D}_2, \mathrm{D}_3)$), the kinetic energy of the spherical motion is

$$\begin{aligned} \frac{\dot{\vec{r}}^2}{2} &= \frac{1}{2X_0}\left(\dot{\vec{\mathrm{D}}} - \frac{\vec{\mathrm{D}}\dot{X}_0}{2X_0}\right)^2 \\ &= \left(\frac{\dot{\vec{\mathrm{D}}}^2}{2X_0} - \frac{\dot{X}_0^2}{8X_0^2}\right). \end{aligned} \tag{A.2}$$

The dot here represents differentiation with respect to canonical time t, which differs from the independent variable u used in equation (2.1), as

$$\mathrm{d}u = X_0 \, \mathrm{d}t. \tag{A.3}$$

As a result the angular velocities $\hat{\omega}$ (in the moving frame), which correspond to the canonical time-derivative $\dot{\mathrm{O}}_1$ of the rotation matrix, differ from the ω (see equations (2.4), (2.22) and (2.23)) by a factor X_0:

$$\hat{\omega}_i = X_0 \omega_i. \tag{A.4}$$

The kinetic energy of rotation is thus expected to be expressed by

$$\tfrac{1}{2} \sum_{i=1}^{3} j_i \hat{\omega}_i \tag{A.5}$$

where the j_i are the components of angular momentum in the moving frame, given by equation (2.22):

$$j_i = \frac{\hat{I}_i^2 \hat{\omega}_i}{X_0 I_i}. \tag{A.6}$$

Lastly, the potential energy term is, according to the expression (2.21) of the integral of energy, $\frac{3}{2} X_0$, i.e.

$$\tfrac{3}{2} \vec{\mathrm{D}}^2.$$

This leads to the following Lagrangian formulation:

$$\mathcal{L} = \left(\frac{\dot{\vec{\mathrm{D}}}^2}{2X_0} - \frac{\dot{X}_0^2}{8X_0^2} \right) + \frac{1}{2X_0} \sum_{i=1}^{3} \left(\frac{\hat{I}_i^2 \hat{\omega}_i^2}{I_i} \right) - \frac{3}{2} \vec{\mathrm{D}}^2 \tag{A.7}$$

(where $X_0 \equiv \vec{\mathrm{D}}^2$; $\dot{X}_0 \equiv 2\vec{\mathrm{D}} \cdot \dot{\vec{\mathrm{D}}}$), from which a Hamiltonian may be derived. Writing down the Euler–Lagrange equations, or the corresponding Hamiltonian equations of motion, the system (2.1) is recovered.

It is worth noting the following values of the terms in the above Lagrangian, in terms of the variables (X_n, Y_n) introduced in the present paper:

$$\begin{aligned}
&\frac{\dot{\vec{\mathrm{D}}}^2}{2X_0} \equiv \frac{X_0 X_{20}}{2} \\
&\frac{\dot{X}_0^2}{8X_0^2} \equiv \frac{X_1^2}{2} \\
&\frac{1}{2X_0} \sum_i \left(\frac{\hat{I}_i^2 \hat{\omega}_i^2}{I_i} \right) \equiv \frac{X_0}{2}(X_2 - X_{20}) \\
&\tfrac{3}{2}\vec{\mathrm{D}}^2 \equiv \tfrac{3}{2} X_0
\end{aligned} \tag{A.8}$$

—therefore the value of the Hamiltonian deduced from $\mathcal{L}$ is precisely $\hat{E}$, given by equation (2.21), as it should be.

Appendix B. Functional independence of the five integrals of motion

As shown in appendix A, the differential system considered here represents Hamiltonian motion in five-dimensional (curved) space, so that five functionally independent integrals of motion ($\hat{E}$, $\vec{J}^2$, J_3, I_6 and L_6) are required for Liouville integrability.

Let us first remark that the integral J_3, whose expression explicitly involves the coordinates of the space O(3), must be independent of all the other integrals. It is thus sufficient to show that $\vec{J}^2$ is independent of $\hat{E}$ (it obviously is), that I_6 is independent of $\hat{E}$ and $\vec{J}^2$ and lastly that L_6 is independent of $\hat{E}$, $\vec{J}^2$ and I_6.

I_6 is at least independent of $\hat{E}$ and $\vec{J}^2$ in the block-diagonal case, since all three integrals may then be given arbitrary values (Gaffet 2000a; see in particular equation (6.6) therein); therefore I_6, $\hat{E}$ and $\vec{J}^2$ cannot be functionally dependent in general.

For the same reason (recalling that L_6 vanishes in block-diagonal cases), L_6 cannot be functionally dependent on $\hat{E}$, $\vec{J}^2$ and I_6 without the function L_6 ($\hat{E}$, $\vec{J}^2$, I_6) being identically zero—which it is not (it is easy to find one point in phase space where L_6 differs from zero: in fact, almost any point will do).

We conclude that the five integrals of the motion are functionally independent.

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